

## Solution of the diffusion equation using Adomain decomposition

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**ABSTRACT:** The objective is estimated the concentration of air pollution, by solving the atmospheric diffusion equation (ADE) using Adomain decomposition method. The solution depends on eddy diffusivity profile (K) and wind speed at the released point (u). We solve the ADE numerically in two dimensions using Adomain decomposition method, then, compared our results with observed data.

**Keywords:** Adomain decomposition/ eddy diffusivity/ atmospheric diffusion equation.

### INTRODUCTION

The Adomian decomposition method (ADM) has been applied in wide class of stochastic and deterministic problems in many interesting mathematics and physics areas (Adomain, 1994). Adomain gave a review of the decomposition method in (Adomain, 1988). Bellomo and Monaco(1985) have used ADM in solving random nonlinear differential equations, Wazwaz (2001) found the numerical solution of sixth order boundary value problem by ADM, El-Sayed and Abdel – Aziz (2003) compared between Adomians decomposition method and wavelet – Galerkin method for solving integro- differential equations. El-gamel (2007) compared between the Sine –Galerkin and the modified decomposition methods for two – point boundary –value problems.

In this paper, advection diffusion equation was solved in two dimensional space (x,z) using Adomian decomposition method to obtain the normalized crosswind integrated concentration employing numerical form. Two forms models of the eddy diffusivities as well as the wind speed at the released point were used in the solution. Two calculated models were compared with observed data measured at Copenhagen in Denmark my using statistical technique.

#### Numerical Method

Time dependent advection – diffusion equation is written as (Arya, 1995)

$$\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} = \frac{\partial}{\partial x} \left( k_x \frac{\partial c}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_y \frac{\partial c}{\partial y} \right) + \frac{\partial}{\partial z} \left( k_z \frac{\partial c}{\partial z} \right) \quad (1)$$

where:

c is the average concentration of air pollution ( $\mu\text{g}/\text{m}^3$ ).

u is the wind speed (m/s).

$K_x$ ,  $k_y$  and  $k_z$  are the eddy diffusivities coefficients along x, y and z axes respectively ( $\text{m}^2/\text{s}$ ).

For steady state, taking  $dc/dt=0$  and the diffusion in the x-axis direction is assumed to be zero compared with the advective in the same directions, hence:

$$u \frac{\partial c}{\partial x} = \frac{\partial}{\partial y} \left( k_y \frac{\partial c}{\partial y} \right) + \frac{\partial}{\partial z} \left( k_z \frac{\partial c}{\partial z} \right) \quad (2)$$

Assuming that  $k_y = k_z = k(x)$ , integrating the equation (2) with respect to y, we obtain the normalized crosswind integrated concentration  $c_y(x, z)$  of contaminant at a point (x, z) of the atmospheric advection–diffusion equation is written in the form (Essa et al. 2006):

$$\frac{\partial^2 c_y(x,z)}{\partial z^2} = \frac{u \partial c_y(x,z)}{K \partial x} \quad (3)$$

Equation (3) is subjected to the following boundary condition

1-It is assumed that the pollutants are absorbed at the ground surface i.e. where  $v_g$  is the deposition velocity (m/s).

$$k \frac{\partial c_y(x,z)}{\partial z} = -v_g c_y(x,z) \quad \text{at } z=0 \quad (i)$$

2-The flux at the top of the mixing layer can be given by

$$k \frac{\partial c_y(x,z)}{\partial z} = 0 \quad \text{at } z=h \quad (ii)$$

3-The mass continuity is written in the form:-

$$u c_y(x,z) = Q \delta(z-h) \quad \text{at } x=0 \quad (iii)$$

where  $\delta$  is Dirac delta function, Q is the source strength and h is mixing height.

4-The concentration of the pollutant tends to zero at large distance of the source, i.e.

$$c_y(x,z) = 0 \quad \text{at } z=\infty \quad (iv)$$

In equation (3), we take  $A=u/K$  and Equation (3) can be solved using Adomain decompositions method as follows:

$$L_{zz} c_y(x,z) = A L_x c_y(x,z)$$

$$\text{where } L_{zz} = \frac{\partial^2}{\partial z^2}, L_x = \frac{\partial}{\partial x}$$

Multiplying both sides of the above equation by  $L_{zz}^{-1}$  (inverse), one gets:

$$c_y(x,z) = c_0 + A L_{zz}^{-1} L_x c_y(x,z) \\ L_{zz}^{-1} = \int_z^z \int_0^z (c_0 + A L_{zz}^{-1} L_x c_y(x,z)) dz dz \quad (4)$$

Assuming that:-

$$C_0 = M(x) + z N(x) \quad (4a)$$

where M and N are unknown functions which will be determined from boundary conditions, using equation (4) to get the general solution in the form:-

$$c_{n+1} = A \int_0^z \int_0^z \frac{\partial c_n}{\partial x} dz dz \quad (5)$$

Put  $n=0$

$$c_1 = A \int_0^z \int_0^z \frac{\partial c_0}{\partial x} dz dz = A \int_0^z \int_0^z \left( \frac{\partial M}{\partial x} + z \frac{\partial N}{\partial x} \right) dz dz = A \frac{\partial M}{\partial x} \frac{z^2}{2!} + A \frac{\partial N}{\partial x} \frac{z^3}{3!} \quad (6)$$

Assuming the solution has the form:-

$$W_n = \sum_{n=0}^{\infty} c_n$$

$$W_1 = c_0 + c_1 = M(x) + z N(x) + A \frac{\partial M}{\partial x} \frac{z^2}{2!} + A \frac{\partial N}{\partial x} \frac{z^3}{3!} \quad (7)$$

By differentiating the equation (7) with respect to z and multiplying by  $k_z$ , we obtain that:

$$k_z \frac{\partial W_1}{\partial z} = k_z N(x) + A z k_z \frac{\partial M}{\partial x} + A \frac{z^2}{2!} k_z \frac{\partial N}{\partial x} \quad (8)$$

Using the boundary condition (i) at  $z=0$ , we obtain that: -

$$k_z \frac{\partial W_1}{\partial z} = k_z N(x) = -v_g M(x)$$

$$\therefore N(x) = \frac{-v_g}{k_z} M(x) \Rightarrow M(x) = -\frac{k_z}{v_g} N(x) \quad (9)$$

Using the boundary condition (ii) at  $z=h$ , we obtain that: -

$$k N(x) + A h k \frac{\partial M}{\partial x} + \frac{h^2}{2!} A k \frac{\partial N}{\partial x} = 0$$

$$\therefore M(x) = \frac{-k}{v_g} N(x)$$

$$\therefore \frac{\partial M(x)}{\partial x} = \frac{-k}{v_g} \frac{\partial N}{\partial x} - \frac{N(x)}{v_g} \frac{\partial k}{\partial x}$$

$$\begin{aligned}
 &k N(x) - A h k \left( \frac{k}{v_g} \frac{\partial N}{\partial x} + \frac{N(x)}{v_g} \frac{\partial k}{\partial x} \right) + \frac{h^2}{2!} A k \frac{\partial N}{\partial x} = 0 \\
 &\left[ \frac{h^2 A k}{2!} - \frac{A h k^2}{v_g} \right] \frac{\partial N}{\partial x} + \left[ k - \frac{A h k}{v_g} \frac{\partial k}{\partial x} \right] N(x) = 0 \Rightarrow \\
 &\left[ \frac{h^2 k A v_g - 2 A h k^2}{2 v_g} \right] \frac{\partial N}{\partial x} = \left[ \frac{A h k \frac{\partial k}{\partial x} - k v_g}{v_g} \right] N(x) \Rightarrow \\
 &\frac{\partial N}{N(x)} = \left( \frac{2 A \frac{\partial k}{\partial x} - 2 v_g}{A h (h v_g - 2 k)} \right) \partial x \tag{9a}
 \end{aligned}$$

Integrating the equation (9a) from 0 to x, we obtain that:-

$$N(x) = N_0(x) e^{\frac{(2 A \frac{\partial k}{\partial x} - 2 v_g)}{h A (h v_g - 2 k)} x} \tag{10}$$

Using the boundary condition (iii), we get that:-

$$N_0(x) = \frac{Q}{u} \delta(z - h)$$

Substituting from  $N_0(x)$  in equation (10), we get that:-

$$N(x) = \frac{Q}{u} \delta(z - h) e^{\frac{2 \left( A h \frac{\partial k}{\partial x} - v_g \right) x}{A h (h v_g - 2 k)}} \tag{11}$$

Substituting from two equations (9) and (11) in equation (4a), we obtain that:-

$$c_0 = \frac{-k}{v_g} N(x) + z N(x) = \left( \frac{-k}{v_g} + z \right) N(x) = (z - B) N(x) \tag{12}$$

Where  $B = k/v_g$

$$\therefore \frac{\partial N}{\partial x} = N \left( \frac{2 \left( A h \frac{\partial k}{\partial x} - v_g \right)}{A h (h v_g - 2 k)} \right) \tag{13}$$

$$M = \frac{-k \partial N}{v_g \partial x}$$

$$\therefore \frac{\partial M}{\partial x} = \frac{-k N}{v_g} \left( \frac{2 \left( A h \frac{\partial k}{\partial x} - v_g \right)}{A h (h v_g - 2 k)} \right) \tag{14}$$

Substituting equations (9a) and (14) in equation (6), we obtain that :-

$$c_1 = (A D) \left( \frac{z^3}{3!} - \frac{k z^2}{v_g 2!} \right) N \tag{15}$$

Where Similar, we get

$$D = N \left( \frac{2 \left( A h \frac{\partial k}{\partial x} - v_g \right)}{A h (h v_g - 2 k)} \right) \tag{16}$$

The general solution:

$$c_2 = (A D)^2 \left( \frac{z^5}{5!} - \frac{k z^4}{v_g 4!} \right) N$$

$$c_3 = (A D)^3 \left( \frac{z^7}{7!} - \frac{k z^6}{v_g 6!} \right) N$$

$$c_4 = (A D)^4 \left( \frac{z^9}{9!} - \frac{k z^8}{v_g 8!} \right) N$$

$$\frac{c_y(x, z)}{Q} = \frac{v_g}{u(hv_g - k)} e^{\frac{2\left(Ah \frac{\partial k}{\partial x} - v_g\right)x}{Ah(hv_g - 2k)}} \left( \sum_0^n \frac{2u\left(Ah \frac{\partial k}{\partial x} - v_g\right)x}{Ahk(hv_g - 2x)} \right)^i \left( \frac{-k z^{2i}}{v_g (2i)!} + \frac{z^{2i+1}}{(2i+1)!} \right) \tag{17}$$

**RESULTS AND DISCUSSION**

We can obtain the wind speed at source height 115m as follows:

$$u_{115} = u_{10} \left(\frac{z}{10}\right)^p \tag{27} \quad \text{Where:-}$$

U115 is the wind speed at 115m.

U10 is the wind speed at 10m height.

z is the physical height.

p is a parameter estimated by Irwin (1979), which is related to stability classes, is given in Table (1).

Table 1. Estimates of the power (p) in urban areas for six Stability Classes based on information by Irwin (1979b)

Stability Classes	Very Unstable (A)	Moderately Unstable (B)	Slightly unstable (C)	Neutral (D)	Slightly stable (E)	Moderately Stable (F)
Urban p	0.19	0.21	0.32	0.30	0.36	0.46

In the present model, we used two methods for the calculation of the eddy diffusivity depends on the downwind distance (x). The first method taking k in the from  $k_1(x) = 0.04ux$  and the second method are referenced to (Arya, 1995) where k takes in the form:-

$$k_z(x) = 0.16 \left( \frac{\sigma_w^2}{u} \right) x$$

Where  $\sigma_w$  is the standard deviation of the vertical velocity.

Table 2. Values of wind speed at 10 m and 115 m and downwind distance through unstable and neutral stabilities in northern part of Copenhagen

Run no.	Stability	u 10 (m/s)	U115 (m/s)	Distance (x) (m)
1	Very unstable (A)	2.1	3.34	1900
1	Very unstable (A)	2.1	3.34	3700
2	Slightly unstable (C)	4.9	10.71	2100
2	Slightly unstable (C)	4.9	10.71	4200
3	Moderately unstable (B)	2.4	4.01	1900
3	Moderately unstable (B)	2.4	4.01	3700
3	Moderately unstable (B)	2.4	4.01	5400
5	Slightly unstable (C)	3.1	4.93	2100
5	Slightly unstable (C)	3.1	4.93	4200
5	Slightly unstable (C)	3.1	4.93	6100
6	Slightly unstable (C)	7.2	11.45	2000
6	Slightly unstable (C)	7.2	11.45	4200
6	Slightly unstable (C)	7.2	11.45	5900
7	Moderately unstable (B)	4.1	6.85	2000
7	Moderately unstable (B)	4.1	6.85	4100
7	Moderately unstable (B)	4.1	6.85	5300
8	Neutral (D)	4.2	8.74	1900
8	Neutral (D)	4.2	8.74	3600
8	Neutral (D)	4.2	8.74	5300
9	Slightly unstable (C)	5.1	11.14	2100
9	Slightly unstable (C)	5.1	11.14	4200
9	Slightly unstable (C)	5.1	11.14	6000

The used data set was observed from the atmospheric diffusion experiments conducted at the northern part of Copenhagen, Denmark, under unstable conditions (Gryning and Lyck, 1984; Gryning et al., 1987). The tracer sulfur hexafluoride (SF6)

Table 3. Comparison between Observed, two numerical models normalized crosswind-integrated concentrations  $C_y/Q$  and downwind distance

un no.	Stability	Down distance (m)	$C_y/Q * 10^{-4}$ (s/m <sup>2</sup> )		Observed
			Numerical model 1	Numerical model 2	
1	Very unstable (A)	1900	3.59	2.08	6.48
1	Very unstable (A)	3700	4.93	3.79	2.31
2	Slightly unstable (C)	2100	7.36	4.03	5.38
2	Slightly unstable (C)	4200	2.04	1.27	2.95
3	Moderately unstable (B)	1900	1.05	1.32	8.2
3	Moderately unstable (B)	3700	8.94	3.40	6.22
3	Moderately unstable (B)	5400	1.20	6.25	4.3
5	Slightly unstable (C)	2100	1.18	3.55	6.72
5	Slightly unstable (C)	4200	1.69	8.75	5.84
5	Slightly unstable (C)	6100	3.76	1.53	4.97
6	Slightly unstable (C)	2000	2.02	2.83	3.96
6	Slightly unstable (C)	4200	1.44	7.24	2.22
6	Slightly unstable (C)	5900	5.31	1.18	1.83
7	Moderately unstable (B)	2000	1.81	2.63	6.7
7	Moderately unstable (B)	4100	1.46	6.09	3.25
7	Moderately unstable (B)	5300	1.01	8.62	2.23
8	Neutral (D)	1900	5.14	7.11	4.16
8	Neutral (D)	3600	9.14	1.50	2.02
8	Neutral (D)	5300	4.32	2.42	1.52
9	Slightly unstable (C)	2100	5.97	3.50	4.58
9	Slightly unstable (C)	4200	1.05	7.70	3.11
9	Slightly unstable (C)	6000	1.60	1.18	2.59

Was released from a tower at a height of 115m without buoyancy. The values of different parameters such as stability, wind speed at 10m (U10), wind speed at 115m (U115), and downwind distance during the experiment are represented in (Table 2).

Table (3); shows the observed, two analytical models, and two numerical normalized crosswind-integrated concentrations  $C_y/Q$  and downwind distance.

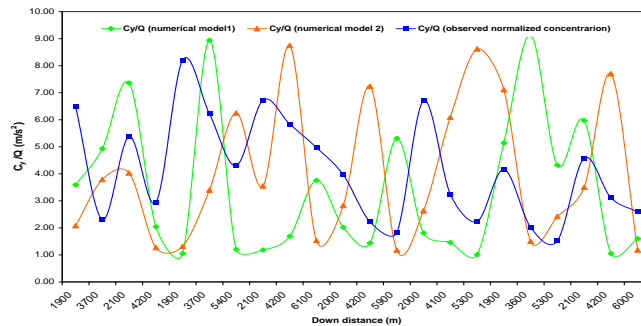


Figure 1. Comparison between numerical cross observed normalized crosswind integrated concentration and downwind distance

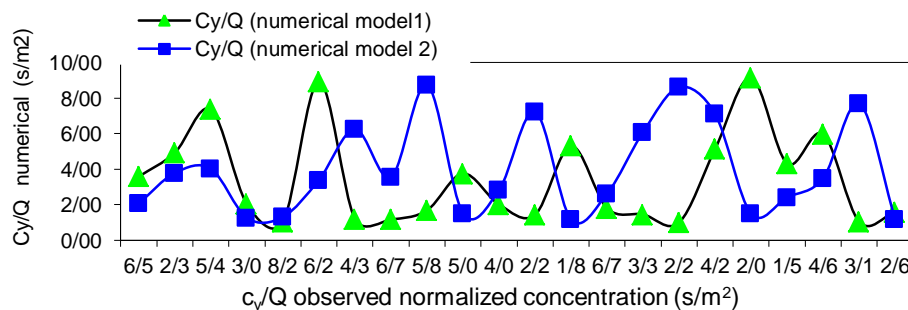


Figure 2. The variation of the numerical predicted normalized crosswind concentrations via observed normalized crosswind concentrations

Fig. (1), shows that the variation of numerical and observed normalized crosswind concentrations data downwind distances. We find that numerical model 1 have points agree with the observed data, while the others points are over predicated.

Fig. (2) Show comparison between numerical model 1, 2 and observed normalized crosswind integrated concentrations. We find that numerical model 1 agree with observed data than numerical model 2, while numerical model 2 has most points are over predicted with the observed data.

**Statistical method**

Now, the statistical method is presented and comparison among analytical, statically and observed results will be offered (Hanna 1989). The following standard statistical performance measures that characterize the agreement between model prediction ( $C_p=C_{pred}/Q$ ) and observation ( $C_o=C_{obs}/Q$ ):

$$\text{Normalized Mean Square Error (NMSE)} = \frac{(C_p - C_o)^2}{(C_p C_o)}$$

$$\text{Fractional Bias (FB)} = \frac{(\overline{C_o} - \overline{C_p})}{[0.5(\overline{C_o} + \overline{C_p})]}$$

$$\text{Correlation Coefficient (COR)} = \frac{1}{N_m} \sum_{i=1}^{N_m} (C_{pi} - \overline{C_p}) \times \frac{(C_{oi} - \overline{C_o})}{(\sigma_p \sigma_o)}$$

$$\text{Factor of Two (FAC2)} = 0.5 \leq \frac{C_p}{C_o} \leq 2.0$$

Where  $\sigma_p$  and  $\sigma_o$  are the standard deviations of  $C_p$  and  $C_o$  respectively. Here the over bars indicate the average over all measurements ( $N_m$ ). A perfect model would have the following idealized performance:  $NMSE = FB = 0$  and  $COR = FAC2 = 1.0$

Table 4. Comparison between our different models according to standard statistical performance measure

Models	NMSE	FB	COR	FAC2
Numerical model 1	0.66	0.04	-0.11	1.19
Numerical model 2	0.79	0.19	-0.08	1.09

From the statistical method, we find that the two models are factors of 2 with observed data. Regarding to NMSE, numerical model 1 is better than numerical model 2. The numerical model 1 is also the best regarding to FB. The correlations of numerical model 1 and model 2 are equal -0.11 and -0.08 respectively.

**CONCLUSION**

We have used numerical solution of two- dimensional atmospheric diffusion equation by Adomain decomposition method to calculate normalized crosswind concentrations for continuous emits sulfur hexafluoride (SF6). In this model the vertical eddy diffusivity depends on the downwind distance and it is calculated using two methods  $k_1(x) = 0.04 u x$  and  $k_2(x) = 0.16 (\sigma_w/u) x$ .

Graphically, we can observe that numerical models 1 and two have most points inside a factor of two with the observed data.

From the statistical method, we find that the two models are factors of 2 (FAC2) Regarding to NMSE, numerical models 1 and two are better with observed data. Also the numerical models 1 and 2 are the best regarding to FB. The correlations of numerical model 1 and model 2 are equal -0.11 and -0.08 respectively.

**REFERENCES**

Wazwaz; The numerical solution of sixth order boundary value problem by the modified decomposition method, Appl. Math. Comput. 118 (2001)311325.  
 ARYA, P.S.Modelling and parameterization of near –source diffusion in weak winds .J.Appl. Met.v.34, p.1112-1122, 1995.  
 Adomain G.; Solving Frontier problems of physics. The Decomposition Method, Kluwer,Boston,(1994).  
 Adomain G.; A review of the decomposition method in applied Mathematics, J.Math.Anal.Appl.135 (1988)501-544.  
 Elgamel M.; Comparison between the SincGalerkin and the modified decomposition methods for solving two- point boundary – value problems, Appl. J. Comput. Phys. 223 (2007) 369-383.

- El-Sayed, M.Abdel-Aziz; Comparison of Adomains decomposition method and wavelet –Galerkin method for solving integrodifferential equation, Appl. Math.Cpoput.139 (2003)151-159.
- Essa, K.S.M., Maha S. and EL-Qtaify, 2006 "Diffusion from a point source in an urban atmosphere", Meteorol.Atmo, Phys., 92, 95-101.
- Hanna, S.R., 1989, "confidence limit for air quality models as estimated by bootstrap and Jacknife resembling methods", Atom. Environ. 23,1385-1395.
- Gryning S. E., and Lyck E., (1984), "Atmospheric dispersion from elevated sources in an urban area: Comparison between tracer experiments and model calculations", J. Climate Appl. Meteor., 23, pp. 651-660.
- Gryning, S.E., Holtslag, A.A.M., Irwin, J.S., Sivertsen, B., (1987), "Applied dispersion modeling based on meteorological scaling parameters", Atmos. Environ. 21 (1), 79-89.
- Irwin, J.S., (1979)." A Theoretical Variation of the wind power law exponent as a function of surface roughness and stability", Atmospheric environment, 13: 191-194.
- Van ULDEN, A.O. (1978). Simple estimates for vertical dispersion from sources near the ground, Atoms. Environ. v.12, p.2125-2129.